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Gravitational Faraday-Effect

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Abstract

The influence of the gravitational field upon the direction of polarization of electromagnetic radiation is generally discussed and for the special case of the gravitational field of a rotating homogeneous ring quantitatively calculated in linear approximation.

The existing proofs of Einstein's general relativistic theory of gravitation by the use of electromagnetic radiation are based on the measurement of frequency-shifts of spectral lines and on the determination of the deflection of light beams or of the retardation of radar signals in the gravitational field of the sun. These tests result exclusively from the *geometrical* optics, because for their derivation only the fact of light propagation along geodesic null-lines of the space-time is needed.

In this connection the question arises if also properties of the *wave* optics are suitable for testing Einstein's theory of gravitation, for instance the polarizability of electromagnetic radiation. Indeed, such gravitational influences upon the polarization of light are to be expected: In view of the Thirring-effect (Thirring, 1918) the plane of polarization of linearly polarized light should be twisted, when the light beam passes through the gravitational field of a *rotating* body (dragging effect according to Mach's principle), analogously to the magnetic Faraday-effect in dispersing media.

In the following, this 'gravitational Faraday-effect' will be discussed explicitly. At first we determine the general behaviour of the direction of polarization along the light beam, and then we calculate the rotation of the plane of polarization in the special case of the gravitational field of a rotating ring. Certainly, the order of magnitude of the rotation angle will not be sufficient for its detection in a terrestial experiment; however, it seems to be not excluded that this effect has a non-negligible importance in cosmic dimensions.

1. *Definition of the Polarization*

We start from the antisymmetric field-tensor $F_{\mu\nu}$ of electromagnetic radiation in form of a light-like bivector (comp. e.g. Kundt, 1961)

$$F_{\mu\nu} = b_\mu k_\nu - b_\nu k_\mu \tag{1.1}$$

with†

$$k_\mu k^\mu = 0, \qquad b_\mu k^\mu = 0, \qquad b_\mu b^\mu < 0 \tag{1.1a}$$

wherein k_μ is the light-like wave vector. From (1.1) the well-known properties of electromagnetic radiation fields‡

$$\begin{array}{lll} F_{\mu\nu} F^{\mu\nu} = 0, & F^*_{\mu\nu} F^{\mu\nu} = 0, & F_{\mu\nu} k^\nu = 0 \\ F^*_{\mu\nu} k^\nu = 0, & F_{\mu\nu} b^\nu \sim k_\mu, F^*_{\mu\nu} b^\nu = 0, & S_{\mu\nu} \sim k_\mu k_\nu \end{array} \tag{1.2}$$

follow and inversely, in which $S_{\mu\nu}$ represents the energy tensor of the radiation field.

Furthermore, we introduce a time-like normalized vector field u_μ ($u_\mu u^\mu = 1$), the trajectories of which we interpret as the world-lines of the physical observers (observer field) measuring the polarization of the radiation (see, for example, Dehnen, 1970). Then the electric and magnetic field-vector observed by the observer field u_μ is given by

$$E_\mu = F_{\mu\nu} u^\nu = b_\mu k_\nu u^\nu - k_\mu b_\nu u^\nu \tag{1.3}$$

and

$$H_\mu = F^*_{\mu\nu} u^\nu$$

respectively, with (comp. (1.2))

$$\begin{array}{llll} E_\mu H^\mu = 0, & E_\mu k^\mu = 0, & H_\mu k^\mu = 0, & E_\mu u^\mu = 0 \\ H_\mu u^\mu = 0, & E_\mu E^\mu < 0, & H_\mu H^\mu < 0 & \end{array} \tag{1.3a}$$

Because of these relations we can restrict ourselves to the discussion of the behaviour of the electric field-vector E_μ for analysing the polarization state. Accordingly, we substitute with the use of (1.3) in (1.1) b_μ by E_μ and get

$$F_{\mu\nu} = E_\mu \frac{k_\nu}{k_\sigma u^\sigma} - E_\nu \frac{k_\mu}{k_\sigma u^\sigma} \tag{1.4}$$

Since only the direction of E_μ characterizes the polarization state, it is appropriate to decompose E_μ into the absolute value E and the unit vector a_μ

$$E_\mu = E a_\mu, \qquad a_\mu a^\mu = -1 \tag{1.4a}$$

Herein E represents the amplitude and a_μ the polarization-vector of the electromagnetic wave.

† We use the following signature of the metric: $-, -, -, +$.
‡ $F^*_{\nu\mu}$ is the dual tensor of $F_{\mu\nu}$.

2. *Propagation of Polarization and Amplitude*

Setting (1.4) and (1.4a) into the vacuum Maxwell-equations†

$$F_{\mu}{}^{\nu}{}_{\parallel\nu} = 0 \tag{2.1a}$$

$$F^{*}_{\mu}{}^{\nu}{}_{\parallel\nu} = 0 \tag{2.1b}$$

one gets the differential equations for the *free* propagation of the polarized electromagnetic wave. In this way it follows with the abbreviation

$$l_{\mu} = \frac{k_{\mu}}{k_{\sigma} u^{\sigma}} \; (l_{\mu} u^{\mu} = 1) \tag{2.2}$$

after a simple calculation from (2.1a)

$$E_{|\nu} l^{\nu} + E(l^{\nu}{}_{\parallel\nu} + l_{\mu\parallel\nu} a^{\mu} a^{\nu}) = 0 \tag{2.3a}$$

$$a_{\mu\parallel\nu} l^{\nu} - l_{\alpha\parallel\beta} a^{\alpha} a^{\beta} a_{\mu} + (u_{\alpha\parallel\beta} - u_{\beta\parallel\alpha}) a^{\alpha} l^{\beta} l_{\mu} - l_{\mu\parallel\nu} a^{\nu} = 0 \tag{2.3b}$$

and from (2.1b)

$$E_{|\nu} l^{\nu} - E(l_{\mu\parallel\nu} a^{\mu} a^{\nu} + u_{\mu\parallel\nu} l^{\mu} l^{\nu}) = 0 \tag{2.4a}$$

$$a_{\mu\parallel\nu} l^{\nu} + l_{\alpha\parallel\beta} a^{\alpha} a^{\beta} a_{\mu} + u_{\alpha\parallel\beta} a^{\alpha} l^{\beta} l_{\mu} - l_{\alpha\parallel\beta} a^{\alpha} u^{\beta} l_{\mu} + l_{\nu\parallel\mu} a^{\nu} = 0 \tag{2.4b}$$

$$l_{\mu\parallel\nu} l^{\nu} = -u_{\alpha\parallel\beta} l^{\alpha} l^{\beta} l_{\mu} \tag{2.4c}$$

Herein the last equation (2.4c) means the propagation of the radiation along geodesic null-lines in non-affine parametric representation. Furthermore, the addition of (2.3a) and (2.4a) results in the propagation equation for the amplitude

$$E_{|\nu} l^{\nu} + \tfrac{1}{2} E(l^{\nu}{}_{\parallel\nu} - u_{\mu\parallel\nu} l^{\mu} l^{\nu}) = 0 \tag{2.5}$$

whereas the addition of (2.3b) and (2.4b) gives:

$$a_{\mu\parallel\nu} l^{\nu} + u_{\alpha\parallel\beta} a^{\alpha} l^{\beta} l_{\mu} + \tfrac{1}{2}[(l_{\nu\parallel\mu} - l_{\mu\parallel\nu}) a^{\nu} - (l_{\lambda\parallel\nu} - l_{\nu\parallel\lambda}) a^{\lambda} u^{\nu} l_{\mu}] = 0 \tag{2.6}$$

The projections of the left side of the last equation upon the wave vector l_{μ} and upon the vector field u_{μ} vanish identically in view of (1.3a) and (2.4c). Therefore, the only non-trivial information from (2.6) is obtained by projection upon the two-dimensional space-like screen-plane of the observers orthogonally oriented to the light rays; this projection will be realized by the symmetric projection tensor (see, for example, Jordan *et al.*, 1961)

$$h_{\mu\nu} = g_{\mu\nu} + l_{\mu} l_{\nu} - l_{\mu} u_{\nu} - l_{\nu} u_{\mu} \tag{2.7}$$

with the properties

$$h_{\mu\nu} h^{\nu}{}_{\lambda} = h_{\mu\lambda}, \quad h_{\mu}{}^{\mu} = 2, \quad h_{\mu\nu} l^{\nu} = 0, \quad h_{\mu\nu} u^{\nu} = 0, \quad h_{\mu\nu} a^{\nu} = a_{\mu} \tag{2.7a}$$

† $\parallel\nu$ signifies the covariant and $|\nu$ the ordinary derivative with respect to the coordinate x^{ν}.

where $g_{\mu\nu}$ is the metric tensor of the space-time. Thus the application of (2.7) to (2.6) results in the following reduced equation of propagation for the direction of polarization

$$a_{\mu\,\|\,\nu}\, l^{\nu}\, h^{\mu}{}_{\alpha} + \tfrac{1}{2}(l_{\nu\,\|\,\mu} - l_{\mu\,\|\,\nu})\, a^{\nu}\, h^{\mu}{}_{\alpha} = 0 \tag{2.8}$$

Finally, the only non-trivial information obtained by subtraction of (2.3a) and (2.4a) or of (2.3b) and (2.4b) is

$$[(l_{\mu\,\|\,\nu} + l_{\nu\,\|\,\mu}) - (l^{\sigma}{}_{\|\,\sigma} + u_{\sigma\,\|\,\lambda}\, l^{\sigma}\, l^{\lambda})\, h_{\mu\nu}]\, a^{\mu}\, a^{\nu} = 0 \tag{2.9}$$

With respect to the arbitrary orientation of a_{μ} in the screen-plane, equation (2.9) is, in view of (2.7a), equivalent to

$$[l_{\mu\,\|\,\nu} + l_{\nu\,\|\,\mu} - (l^{\sigma}{}_{\|\,\sigma} + u_{\sigma\,\|\,\lambda}\, l^{\sigma}\, l^{\lambda})\, h_{\mu\nu}]\, h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta} = 0 \tag{2.10}$$

For the following it is useful to substitute in the propagation equations (2.4c), (2.5), (2.8) and (2.10) the wave vector l_{μ} again by k_{μ} according to (2.2); in this way we get successively:

$$k_{\mu\,\|\,\nu}\, k^{\nu} = 0 \tag{2.11}$$

$$E_{|\nu}\, k^{\nu} = -\tfrac{1}{2} E k^{\nu}{}_{\|\,\nu} + E u_{\mu\,\|\,\nu} \frac{k^{\mu}\, k^{\nu}}{k_{\sigma}\, u^{\sigma}} \tag{2.12}$$

$$a_{\alpha\,\|\,\nu}\, k^{\nu}\, h^{\alpha}{}_{\mu} = \tfrac{1}{2}(k_{\alpha\,\|\,\nu} - k_{\nu\,\|\,\alpha})\, a^{\nu}\, h^{\alpha}{}_{\mu} \tag{2.13}$$

$$(k_{\mu\,\|\,\nu} + k_{\nu\,\|\,\mu} - k^{\sigma}{}_{\|\,\sigma}\, h_{\mu\nu})\, h^{\mu}{}_{\alpha}\, h^{\nu}{}_{\beta} = 0 \tag{2.14}$$

Equation (2.11) means the propagation of the radiation along geodesic null-lines, now in affine parametric representation [cf. equation (2.4c)]. The equations (2.12) and (2.13) describe the propagation of the amplitude E and the polarization a_{μ}, according to which in the first case especially the *expansion* (divergence) and in the second one the *rotation* of the light rays has a considerable influence upon the behaviour of E and a_{μ} respectively. Finally, equation (2.14) represents the well-known fact that the rays of electromagnetic radiation show no shearing.

Accordingly for the discussion of the gravitational Faraday-effect it is necessary to solve at first the differential equation (2.11) for the geodesic null-lines and then to determine the behaviour of the polarization a_{μ} along the rays by integration of equation (2.13). Because in case of monochromatic radiation $k_{\sigma} u^{\sigma} = \nu$ is the frequency observed by the observers u_{μ} and evidently ν does *not* appear in (2.11) and (2.13), in contrast to (2.12), the behaviour of the polarization is independent of the frequency of the radiation. Thus the gravitational Faraday-effect is *free* of dispersion in opposition to the magnetic one.

3. *Gravitational Field of a Uniformly Rotating Homogeneous Ring in Linear Approximation*

In view of Mach's principle [cf. also equation (2.13)], a rotation of the plane of polarization is to be expected, when linearly polarized radiation passes through the gravitational field of a rotating body. For detecting this effect unambiguously 'scattering experiments' are suitable, in which the radiation coming from the flat space-like infinity runs out to the flat infinity after passing through the gravitational field of the rotating body. A simple experiment of this kind is the propagation of linearly polarized radiation along the axis of a rotating thin ring (see Fig. 1).

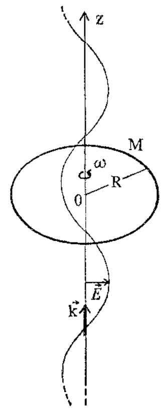

Figure 1.—Propagation of a linearly polarized electromagnetic wave along the axis of a rotating thin ring. A rotation of the plane of polarization is to be expected in the direction of the rotation of the ring.

Therefore, we determine in the following at first the gravitational field of a rotating homogeneous thin ring in the neighbourhood of its axis in linear approximation. In Cartesian coordinates the energy-momentum-stress tensor of an homogeneous ring rotating uniformly in the x, y-plane around the z-axis is given by (Frehland, 1971)

$$T_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & +\rho_0 c\omega y \\ 0 & 0 & 0 & -\rho_0 c\omega x \\ 0 & 0 & 0 & 0 \\ +\rho_0 c\omega y & -\rho_0 c\omega x & 0 & \rho_0 c^2(1+\beta^2) \end{pmatrix} \tag{3.1}$$

with

$$x = R\cos\varphi, \qquad y = R\sin\varphi, \qquad \beta = \frac{\omega R}{c} \tag{3.1a}$$

wherein φ is the angle of rotation around the z-axis, $\omega = \dot{\varphi}$ the angular-velocity, R the radius and ρ_0 the (constant) rest mass density of the ring. After the expansion of the metric of the space-time

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \gamma_{\mu\nu} + \dots, \\ \gamma_{\mu\nu} &= \psi_{\mu\nu} - \tfrac{1}{2}\psi\eta_{\mu\nu}, \end{aligned} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & 0 & \\ & 0 & -1 & \\ & & & +1 \end{pmatrix} \tag{3.2}$$

the integration of the linearized field equations of gravitation

$$\psi_{\mu\nu}{}^{|\sigma}{}_{|\sigma} = -2\kappa T_{\mu\nu} \tag{3.3}$$

($\kappa = 8\pi G/c^4$, G Newton's gravitational constant) yields with the use of (3.1)

$$\begin{aligned}
\psi_{4x} &= -\frac{\kappa c}{2\pi}\omega \int \frac{\rho_0 y'}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
\psi_{4y} &= \frac{\kappa c}{2\pi}\omega \int \frac{\rho_0 x'}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
\psi_{44} &= -\frac{\kappa c^2}{2\pi}\int \frac{\rho_0(1+\beta^2)}{|\mathbf{r}-\mathbf{r}'|}d\tau' \\
\psi_{\mu\nu} &= 0 \text{ otherwise}
\end{aligned} \tag{3.4}$$

Expanding the integrands in (3.4) for the neighbourhood of the z-axis ($x/R \ll 1, y/R \ll 1$) we obtain by calculation of the integrals under the restriction to low velocities (linearized in β):

$$\begin{aligned}
\psi_{44} &= -\frac{\kappa c^2}{2\pi}\frac{M}{\sqrt{(R^2+z^2)}}\left\{1+\frac{1}{4}\frac{(R^2-2z^2)(x^2+y^2)}{(R^2+z^2)^2}+0((x/R)^3,(y/R)^3)\right\} \\
\psi_{4x} &= -\frac{\kappa c}{4\pi}MR^2\omega\frac{y}{(R^2+z^2)^{3/2}}+0((x/R)^3,(y/R)^3) \\
\psi_{4y} &= \frac{\kappa c}{4\pi}MR^2\omega\frac{x}{(R^2+z^2)^{3/2}}+0((x/R)^3,(y/R)^3) \\
\psi_{\mu\nu} &= 0 \text{ otherwise}
\end{aligned} \tag{3.5}$$

wherein M is the total mass of the ring. Herewith we find according to (3.2) the following non-vanishing deviations from the metric of the flat space-time in the neighbourhood of the rotation axis (z-axis) of the ring

$$\begin{aligned}
\gamma_{11} = \gamma_{22} = \gamma_{33} = \gamma_{44} &= -\frac{\kappa c^2}{4\pi}\frac{M}{\sqrt{(R^2+z^2)}}\left\{1+\frac{1}{4}\frac{(R^2-2z^2)(x^2+y^2)}{(R^2+z^2)^2}+\cdots\right\} \\
\gamma_{4x} &= -\frac{\kappa c}{4\pi}MR^2\omega\frac{y}{(R^2+z^2)^{3/2}}+\cdots \\
\gamma_{4y} &= \frac{\kappa c}{4\pi}MR^2\omega\frac{x}{(R^2+z^2)^{3/2}}+\cdots
\end{aligned} \tag{3.6}$$

Evidently only the components γ_{4x} and γ_{4y} result from the rotation ω; the remaining components of $\gamma_{\mu\nu}$ describe the static gravitational field of the mass M of the ring near the z-axis.

4. *Integration of the Propagation Equations*

For the propagation of an electromagnetic wave along the rotation axis of the ring (z-axis) the integration of the differential equation for the geodesic null-lines (2.11) gives under the initial condition at $z = -\infty$

$$\overset{0}{k}{}^{\mu} = (0, 0, 1, 1) \tag{4.1}$$

within the approximations mentioned above

$$k_1 = -\frac{\kappa c^2}{8\pi} M \frac{xz}{(R^2+z^2)^{3/2}}, \qquad k_2 = -\frac{\kappa c^2}{8\pi} M \frac{yz}{(R^2+z^2)^{3/2}},$$
$$k_3 = -\frac{\kappa c^2}{4\pi} \frac{M}{\sqrt{(R^2+z^2)}} \left[1 + \frac{1}{4} \frac{(R^2-2z^2)(x^2+y^2)}{(R^2+z^2)^2}\right] - 1 \tag{4.2}$$
$$k_4 = 1$$

Herewith

$$k_{\mu \| \nu} - k_{\nu \| \mu} = 0 \tag{4.3}$$

is valid (in the neighbourhood of the z-axis) and the propagation equation for the polarization a_μ (2.13) takes the form

$$a_{\alpha|\nu} \overset{0}{k}{}^{\nu} \overset{0}{h}{}^{\alpha}{}_{\mu} = \Gamma^{\sigma}_{\alpha\lambda} \overset{0}{a}_{\sigma} \overset{0}{k}{}^{\lambda} \overset{0}{h}{}^{\alpha}{}_{\mu} \tag{4.4}$$

Within the linear approximation the quantities with the index null are to be considered in the lowest order undisturbed by the gravitational field.

Choosing the observer field u_μ at rest relatively to the inertial system (fixed stars) the following relations hold

$$\overset{0}{u}{}^{\mu} = \delta_4{}^{\mu} \tag{4.5a}$$

and with regard to (2.2), (2.7) and (4.1):

$$\overset{0}{h}{}^{1}{}_{1} = \overset{0}{h}{}^{2}{}_{2} = 1, \qquad \overset{0}{h}{}^{\mu}{}_{\nu} = 0 \text{ otherwise} \tag{4.5b}$$

Furthermore, we assume as initial condition for the polarization at $z = -\infty$

$$\overset{0}{a}{}^{\mu} = \delta_1{}^{\mu} \tag{4.5c}$$

which means linear polarization of the radiation in the direction of the x-axis. Then the equation (4.4) results after calculation of the Christoffel-symbols $\Gamma^{\sigma}_{\alpha\lambda}$ according to (3.6) in:

$$\frac{da_1}{dz} = \tfrac{1}{2}\gamma_{11|z}, \qquad \frac{da_2}{dz} = -\frac{\kappa c}{4\pi} M R^2 \omega (R^2+z^2)^{-3/2} \tag{4.6}$$

In view of (3.6) and (4.5c) we find by integration of (4.6) along the z-axis from $z = -\infty$ up to $z = +\infty$:

$$a_{1(z=+\infty)} = -1$$
$$a_{2(z=+\infty)} = -\frac{\kappa c}{2\pi} M\omega \tag{4.7}$$

The comparison of (4.5c) and (4.7) shows that as a consequence of passing through the gravitational field of the *rotating* ring, the polarization plane of the radiation is swivelled through the (in general small) angle

$$\delta = \frac{\kappa c}{2\pi} M\omega \tag{4.8}$$

in the direction of the rotation of the ring. This is in agreement with Mach's principle, according to which just such a dragging effect should exist. From here it follows immediately that the rotation of the plane of polarization does not be cancelled when the radiation passes subsequently through the gravitational field in opposite direction, for instance as a result of reflection $z = +\infty$, but it will be *doubled* analogously to the situation in case of the magnetic Faraday-effect.

With the substitution $M = 2\pi\sigma R$, wherein σ is the linear mass-density of the ring, equation (4.8) becomes (in CGS-units):

$$\delta = \kappa c^2 \beta\sigma = 1.8 \cdot 10^{-27} \beta\sigma \tag{4.9}$$

The smallness of the factor in (4.9) means that an experimental proof of (4.9) in a terrestrial laboratory seems to be hopeless. However, it cannot be excluded that the gravitational Faraday-effect has a non-negligible importance in cosmic dimensions, where the rotating masses M [cf. equation (4.8)] could become large enough; but in this case the independence of the rotation angle δ on the frequency ν of the radiation (cf. Section 2) should complicate at least its experimental detection.

References

Dehnen, H. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 6, p. 509.
Frehland, E. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 1, p. 33.
Jordan, P., Ehlers, J. and Sachs, R. (1961). *Akad. d. Wiss. u. Lit. Mainz, Math.-Nat. Klasse*, No. 1, p. 12.
Kundt, W. (1961). *Akad. d. Wiss. u. Lit. Mainz, Math.-Nat. Klasse*, No. 3, p. 88.
Thirring, H. (1918). *Physikalische Zeitschrift*, **19**, 33.